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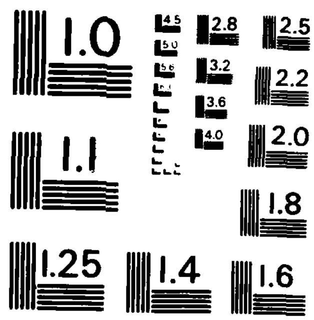

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Computing the Reliability of k out of n Systems

by

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by

Philip J. Boland and Frank Proschan

Abstract

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A-1

1. Introduction.

A system of n components which functions if and only if at least k of the components function is called a k out of n system. Parallel systems are 1 out of n systems, fail safe systems are n-1 out of n systems, and series systems are n out of n systems. We make the assumption that the n components in the system operate independently of one another. Therefore if $\underline{p} = (p_1, \ldots, p_n)$ is the vector of component reliabilities and $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ represents any vector with components equal to zeroes or ones,

then $$h_k(\underline{p}) = \sum_{\underline{\varepsilon},\ \varepsilon_1+\ldots+\varepsilon_n \geq k} p_1^{\varepsilon_1} \cdots p_n^{\varepsilon_n} (1-p_1)^{1-\varepsilon_1} \cdots (1-p_n)^{1-\varepsilon_n}$$

is the probability that k or more of the components function. We refer to $h_k(\underline{p})$: $[0, 1]^n \to [0, 1]$ as the reliability function of a k out of n system with independent components. For parallel, failsafe, and series systems, the reliability function $h_k(\underline{p})$ is a quite manageable function, even for large values of n. In general, however, the behavior of the function $h_k(\underline{p})$ on $[0, 1]^n$ can be quite complex, and the calculation of $h_k(\underline{p})$ at a single vector of component reliabilities can be quite cumbersome. In the relatively simple case of a 5 out of 8 system for example, evaluating $h_5(\underline{p})$ would usually involve calculating 93 products of 8 numbers. In this expository paper we survey some of the important theoretical results which have been obtained about the reliability function $h_k(\underline{p})$, and indicate how these may be used to reduce calculations and obtain good bounds for the reliability of a k out of n system.

2. Basic Theoretical Results.

Hoeffding (1956) considers the problem of finding the maximum and minimum of $h_k(p_1, \ldots, p_n)$ subject to the constraint that $\sum_{i=1}^{n} p_i$ is held fixed. His results are presented in terms of the number of successes in independent Bernoulli trials.

It is clear however that if S is the number of successes in n independent Bernoulli trials with respective success probabilities $p_1, \ldots, p_n$, then Prob$(S \geq k) = h_k(p_1, \ldots, p_n)$. For a given vector $\underline{p} = (p_1, \ldots, p_n)$ of component probabilities, we let $\bar{p} = \sum_{i=1}^{n} p_i/n$, $[\sum_{i=1}^{n} p_i]$ be the greatest integer less than or equal to $\sum_{i=1}^{n} p_i$, and $(1, \ldots, 1, \sum_{i=1}^{n} p_i - [\sum_{i=1}^{n} p_i], 0, \ldots, 0)$ be the vector with $[\sum p_i]$ of its n coordinates equal to 1. Hoeffding proves in particular the following theorem:

Theorem 1. For any vector $\underline{p} = (p_1, \ldots, p_n)$ of component probabilities we have:

$$1 = h_k\left(1, \ldots, 1, \sum_{i=1}^{n} p_i - \left[\sum_{i=1}^{n} p_i\right], 0, \ldots, 0\right) \geq h_k(p_1, \ldots, p_n) \geq h_k(\bar{p}, \ldots, \bar{p}) \quad \text{if } \sum_{i=1}^{n} p_i \geq k,$$

while

$$0 = h_k\left(1, \ldots, 1, \sum_{i=1}^{n} p_i - \left[\sum_{i=1}^{n} p_i\right], 0, \ldots, 0\right) \leq h_k(p_1, \ldots, p_n) \leq h_k(\bar{p}, \ldots, \bar{p}) \quad \text{if } \sum_{i=1}^{n} p_i \leq k-1.$$

Hoeffding also establishes upper and lower bounds for $h_k(\underline{p})$ for the case when $k-1 < \sum_{i=1}^{n} p_i < k$; however these bounds are considerably more complicated.

We present the first of four figures in an attempt to give some geometrical insight into the behavior of $h_k(\underline{p})$. For any α such that $0 \leq \alpha \leq n$, we define $\lambda_\alpha = \{\underline{p}: \sum_{i=1}^{n} p_i = \alpha \text{ and } 0 \leq p_i \leq 1 \text{ for } i=1,\ldots, n\}$. Although the convex set λ_α = is the intersection of the hyperplane $\{\underline{x}: \sum_{i=1}^{n} x_i = \alpha\}$ with the n-dimensional unit cube, it will be represented geometrically by a "line" in the following figures. The n-dimensional cube itself is represented by a "square", and by the "diagonal" we will mean the line of points all of whose coordinates are equal. Theorem 1 says that $h_k(\underline{p})$ restricted to λ_α assumes a maximum of one at the 'extremities' of λ_α and a minimum of $h_k(\alpha/n, \ldots, \alpha/n)$ along the diagonal whenever $\alpha \geq k$, while it assumes a maximum of $h_k(\alpha/n, \ldots, \alpha/n)$ on the diagonal and a minimum of 0 at the extremities of λ_α whenever $\alpha \leq k-1$.

(Figure 1 goes here.)

The theory of majorization and Schur functions provides an elegant format for presenting many of the results concerning the reliability function $h_k(\underline{p})$. Given a vector $\underline{x} = (x_1, \ldots, x_n)$, let $x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[n]}$ denote a decreasing rearrangement of $x_1, \ldots, x_n$: (we write $\underline{x} >^m \underline{y}$) if

$$\sum_{i=1}^{j} x_{[i]} \geq \sum_{i=1}^{j} y_{[i]} \quad \text{for } j=1, \ldots, n-1$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.$$

Hardy, Littlewood and Pólya (1952) show that $\underline{x} >^m \underline{y}$ if and only if there exists a doubly stochastic matrix Π such that $\underline{y} = \underline{x}\Pi$. If $\underline{x} >^m \underline{y}$, then the coordinates of $\underline{x}$ are more "dispersed' than those of $\underline{y}$ (See Marshall and Olkin (1979) for an excellent treatment of majorization.) Schur functions are real valued functions which are monotone with respect to the partial ordering of majorization. A function h with the property that $\underline{x} >^m \underline{y} \Rightarrow h(\underline{x}) \geq (\leq)\ h(\underline{y})$ is called <u>Schur convex</u> (<u>Schur concave</u>). A useful characterization of Schur convexity (-concavity) is given by the Schur-Ostrowski condition, which states that a differentiable permutation invariant function h defined on R^n is Schur convex (Schur concave) if and only if

$$(x_i - x_j)\left(\frac{\partial h}{\partial x_i} - \frac{\partial h}{\partial x_j}\right) \geq (\leq)\ 0 \quad \text{for all } i, j \text{ and } \underline{x} \in R^n.$$

Some useful comparisons for $h_k(\underline{p})$ can be obtained by working with the hazard transform. The hazard transform z_k: $[0, \infty)^n \to [0, \infty)$ of h_k is defined by

$$z_k(R_1, \ldots, R_n) = -\log h_k(e^{-R_1}, \ldots, e^{-R_n}).$$

Pledger and Proschan (1971) prove the following theorem about z_k:

Theorem 2. The hazard transform z_k of a k out of n system is increasing and Schur concave on $[0, \infty)^n$.

A particularly interesting corollary of Theorem 2 is that $h_k(p_1, \ldots, p_n) \geq h_k(p_G, \ldots, p_G)$ where p_G is the geometric mean $(\prod_{i=1}^n p_i)^{1/n}$. Together with Theorem 1 this implies that

$$h_k(\bar{p}, \ldots, \bar{p}) \geq h_k(p_G, \ldots, p_G)$$

whenever $\sum_{i=1}^n p_i \leq k-1.$

Gleser (1975) uses majorization to obtain refinements of the inequalities established by Hoeffding. If S is the number of successes in n independent trials with respective success probabilities $p_1, \ldots, p_n$, then Samuels (1965) shows that Prob(S=j) is unimodal with mode at $[\sum_{i=1}^n p_i]$ or $[\sum_{i=1}^n p_i+1]$. Using this fact together with the Schur-Ostrowski condition, Gleser proves the following theorem:

Theorem 3. The reliability function $h_k(\underline{p})$: $[0, 1]^n \rightarrow [0, 1]$ is Schur convex in the region where $\sum_{i=1}^n p_i \geq k+1$ and Schur concave in the region where $\sum_{i=1}^n p_i \leq k-2$.

Figure 2 may help to illustrate Theorem 3. The region shaded by vertical lines indicates where $h_k(\underline{p})$ is Schur convex, while the area shaded by horizontal lines indicates where $h_k(\underline{p})$ is Schur concave.

(Figure 2 goes here.)

If $\underline{p} = (p_1, \ldots, p_n)$ where $n>2$, we denote by $\underline{p}^{ij}$ the vector in $[0, 1]^{n-2}$ obtained by deleting the i^{th} and j^{th} coordinates of $\underline{p}$. For any $r\geq 0$, we let $h_r^*(\underline{p}^{ij})$ denote the probability that exactly r of the n-2 components (with respective probabilities given by $\underline{p}^{ij}$) function. Boland and Proschan (1983) prove that

$$h_{k-2}^*(\underline{p}^{ij}) \leq (\geq)\ h_{k-1}^*(\underline{p}^{ij}) \qquad \text{for } i\neq j,\ k\geq 2$$

whenever

$p_\ell \geq (\leq) \frac{k-1}{n-1}$ for all $\ell\neq i,j$. This result, coupled with the Schur-Ostrowski condition, enable Boland and Proschan to prove the following Schur property of $h_k(\underline{p})$:

<u>Theorem 4.</u> The reliability function $h_k(\underline{p})$: $[0, 1]^n \rightarrow [0, 1]$ is Schur convex in the region $[\frac{k-1}{n-1}, 1]^n$ and Schur concave in the region $[0, \frac{k-1}{n-1}]^n$.

The point $(\frac{k-1}{n-1}, \ldots, \frac{k-1}{n-1})$ on the diagonal is a focal point in distinguishing the regions where $h_k(\underline{p})$ is Schur convex and Schur concave. The number $\frac{k-1}{n-1}$ plays an important role in the study of the function of one variable $h_k(p) = \sum_{i=1}^{n} \binom{n}{i} p^i(1-p)^{n-i}$. (Note that $h_k(p)$ is the restriction of $h_k(\underline{p})$ to the diagonal.) $h_k(p)$ is convex on the interval $[0, \frac{k-1}{n-1}]$ and concave on the interval $[\frac{k-1}{n-1}, 1]$ (See Barlow and Proschan (1965)). Figure 3 may help to interpret Theorem 4. Again the vertically (horizontally) shaded region indicates where $h_k(\underline{p})$ is Schur convex (Schur concave). Figure 4 illustrates the combined results of Theorem 3 and Theorem 4.

(Figure 3 and Figure 4 go here.)

3. Applications.

We now give a number of examples in order to demonstrate how the properties of $h_k(\underline{p})$ presented can help us both in comparing various k out of n systems and in calculating bounds for the reliability of specified k out of n systems.

Example 1. Let us consider a 3 out of 4 system. Theorem 4 in particular implies that a system with component reliabilities (.7, .8, .9, 1.0) is superior (has higher reliability) than a system with component reliabilities (.75, .75, .95, .95) which in turn is superior to one with component reliabilities (.85, .85, .85, .85). Note that for all three of these systems the four component reliabilities sum to 3.4, and that each component reliability exceeds $\frac{k-1}{n-1} = \frac{2}{3}$. Theorem 4 also implies that the system with component reliabilities (.2, .3, .5, .6) is inferior to one with component reliabilities (.2, .4, .4, .6) which in turn is inferior to one with component reliabilities (.4, .4, .4, .4).

Example 2. We now consider evaluating the reliability of a 5 out of 8 system. As previously indicated, even for this example where k and n are rather small, the standard method for evaluating $h_5(p_1, \ldots, p_8)$ would normally involve (if for example the eight component probabilities are distinct) calculating the sum of 93 products of 8 numbers each. We will now indicate by judiciously 'averaging' some (or all) of the component probabilities, how more easily calculable bounds may be computed for the reliability of the system.

Given a vector $\underline{p} = (p_1, \ldots, p_8)$ of component probabilities, we lose no generality in assuming that $p_1 \le p_2 \le \ldots \le p_8$. Let us use the following notation: $p_{ij} = (p_i + p_j)/2$ and $p_{ij\ell m} = (p_i + p_j + p_\ell + p_m)/4$ for any integers i, j, ℓ, m between 1 and 8. As before $\bar{p}$ and p_G will denote respectively the arithmetic and geometric means of the components of the vector $\underline{p} = (p_1, \ldots, p_8)$.

Note that $\underline{p} = (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) >^m (p_{12}, p_{12}, p_{34}, p_{34}, p_{56}, p_{56}, p_{78}, p_{78})$

$$>^m (p_{1234}, p_{1234}, p_{1234}, p_{1234}, p_{5678}, p_{5678}, p_{5678}, p_{5678})$$

$$>^m (\bar{p}, \bar{p}, \bar{p}, \bar{p}, \bar{p}, \bar{p}, \bar{p}, \bar{p}) = \bar{\underline{p}}$$

The calculation of $h_k(\underline{p})$ becomes easier as the number of distinct component values in $\underline{p}$ decreases. While the calculation of $h_5(\underline{p})$ could involve adding 93 products, the calculations of $h_5(p_{12}, p_{12}, p_{34}, p_{34}, p_{56}, p_{56}, p_{78}, p_{78})$, $h_5(p_{1234}, p_{1234}, p_{1234}, p_{1234}, p_{5678}, p_{5678}, p_{5678}, p_{5678})$ and $h_5(\bar{\underline{p}})$ would involve adding respectively at most 31, 10, and 4 products. If the vector components of $\underline{p} = (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8)$ are all $\geq \frac{k-1}{n-1} = \frac{4}{7}$ (respectively $\leq \frac{4}{7}$), then we can easily determine some lower (upper) bounds for $h_5(\underline{p})$ by using Theorems 1 and 4. To illustrate the accuracy of these bounds we compute some specific probabilities.

a) Let $\underline{p} = (.60, .64, .70, .74, .80, .84, .90, .94)$. Each $p_i \geq \frac{4}{7}$ and we observe that

$$\begin{aligned} h_5(\underline{p}) &= .92097 = h_5(.60, .64, .70, .74, .80, .84, .90, .94) \\ &\geq .92069 = h_5(.62, .62, .72, .72, .82, .82, .92, .92) \\ &\geq .91896 = h_5(.67, .67, .67, .67, .87, .87, .87, .87) \\ &\geq .91201 = h_5(.77, .77, .77, .77, .77, .77, .77, .77). \end{aligned}$$

b) Let $\underline{p} = (.61, .62, .63, .64, .65, .66, .67, .68)$. Then again each $p_i \geq \frac{4}{7}$ and we observe that

$$\begin{aligned} h_5(\underline{p}) &= .69580 = h_5(.61, .62, .63, .64, .65, .66, .67, .68) \\ &\geq .69580 = h_5(.615, .615, .635, .635, .655, .655, .675, .675) \\ &\geq .69576 = h_5(.625, .625, .625, .625, .665, .665, .665, .665) \\ &\geq .69562 = h_5(.645, .645, .645, .645, .645, .645, .645, .645) \end{aligned}$$

c) If $\underline{p} = (.10, .14, .20, .24, .30, .34, .40, .44)$, then each component $p_i \leq \frac{4}{7}$ and we observe that

$$
\begin{aligned}
h_5(\underline{p}) = .03188 &= h_5(.10, .14, .20, .24, .30, .34, .40, .44) \\
.03206 &= h_5(.12, .12, .22, .22, .32, .32, .42, .42) \\
.03318 &= h_5(.17, .17, .17, .17, .37, .37, .37, .37) \\
.03768 &= h_5(.27, .27, .27, .27, .27, .27, .27, .27).
\end{aligned}
$$

Note that for the given vector $\underline{p}$, the geometric mean of the coordinates is $p_G = .24276$. Using the corollary to Theorem 2 we can compute the lower bound for $h_k(\underline{p})$ which is given by $h_5(\underline{p}_G) = .02410$.

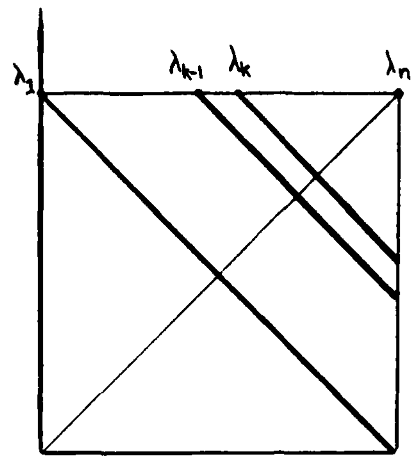

Figure 1

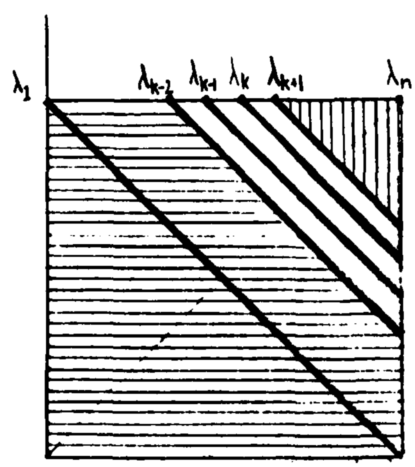

Figure 2

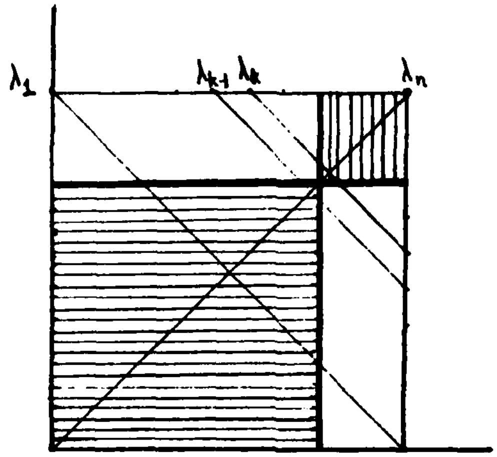

Figure 3

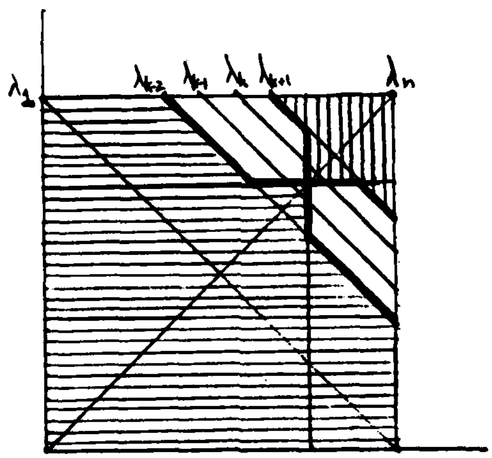

Figure 4

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